

REALIZING UNIFORMLY RECURRENT SUBGROUPS

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ABSTRACT. We show that every uniformly recurrent subgroup of a locally compact group is the family of stabilizers of a minimal action on a compact space. More generally, every closed invariant subset of the Chabauty space is the family of stabilizers of an action on a compact space on which the stabilizer map is continuous everywhere. This answers a question of Glasner and Weiss. We also introduce the notion of a universal minimal flow relative to a uniformly recurrent subgroup and prove its existence and uniqueness.

1. INTRODUCTION

Let G be a locally compact group. Consider the space of subgroups $\text{Sub}(G)$ endowed with the Chabauty topology [C] and recall that a subbasis of open sets for this topology is given by sets of the form

$$\mathcal{U}_C = \{H \in \text{Sub}(G) : H \cap C = \emptyset\} \quad \text{and} \quad \mathcal{U}_V = \{H \in \text{Sub}(G) : H \cap V \neq \emptyset\},$$

where C varies among the compact subsets and V among the open subsets of G . This topology makes $\text{Sub}(G)$ a compact Hausdorff space on which G acts continuously by conjugation.

Glasner and Weiss [GW] initiated the study of *uniformly recurrent subgroups* (URS for short), i.e., closed, invariant, minimal subsets of $\text{Sub}(G)$. This notion can be seen as a topological analogue of the measure-theoretic one of *invariant random subgroup* [AGV]. URSs have recently attracted some attention as it turned out that this notion is a convenient tool to study *boundary actions*, which for discrete groups are connected to C^* -simplicity [K2, LBMB].

As was shown by Glasner and Weiss [GW], a URS is naturally associated to every minimal action $G \curvearrowright X$ on a compact space. Namely, consider the stabilizer map $\text{Stab} : X \rightarrow \text{Sub}(G)$. This map is usually not continuous. However it is upper semi-continuous, in the sense that for every net (x_i) converging to $x \in X$, every cluster point of $\text{Stab}(x_i)$ in $\text{Sub}(G)$ is contained in $\text{Stab}(x)$. This property is enough to ensure that the closure of the image of Stab contains a unique URS. (This result is proved in [GW, Proposition 1.2]. See also the argument of [AG, Lemma I.1] to avoid the assumption, made throughout [GW], that X is metrizable.) The unique URS contained in $\overline{\text{Stab}(X)}$ is called the *stabilizer URS* of $G \curvearrowright X$ and will be denoted by $\mathcal{S}_G(X)$.

In analogy with what is known for IRSs, Glasner and Weiss ask whether every URS arises in this way. In this paper, we answer this question in the affirmative.

Theorem 1.1. *Let G be a locally compact group and let $\mathcal{H} \subseteq \text{Sub}(G)$ be a closed, invariant subset. Then there exists a continuous action of G on a compact space X such that the stabilizer map $\text{Stab} : X \rightarrow \text{Sub}(G)$ is everywhere continuous and its image is equal to \mathcal{H} . If G is second countable, X can be chosen to be metrizable.*

In the above result, \mathcal{H} is not assumed to be a URS, and therefore the action $G \curvearrowright X$ cannot always be minimal. However if \mathcal{H} is a URS, the continuity of Stab

implies that for every minimal invariant subset Y of X , the image $\text{Stab}(Y)$, being a closed minimal invariant subset of \mathcal{H} , is equal to \mathcal{H} . Restricting the action to Y yields the following.

Corollary 1.2. *Every URS of a locally compact group arises as the stabilizer URS of a minimal action on a compact space.*

If $\mathcal{H} = \{\{1_G\}\}$, Theorem 1.1 recovers a classical theorem in topological dynamics, due to Veech [V] (and previously to Ellis [E2] for discrete groups), stating that every locally compact group admits a free action on a compact space. The proof of Theorem 1.1 is largely inspired by the proof of this result.

In addition, given a locally compact group G and a URS $\mathcal{H} \subseteq \text{Sub}(G)$, we define and study a relative universal minimal flow $M(G, \mathcal{H})$ as the largest minimal compact G -space in which every subgroup in \mathcal{H} fixes a point. For $\mathcal{H} = \{\{1_G\}\}$, this is the usual universal minimal flow of G . We prove that $M(G, \mathcal{H})$ is unique up to isomorphism and characterize under what conditions it is metrizable.

Related work. In an independent work [E] that appeared while this paper was being completed, G. Elek proves Corollary 1.2 for finitely generated groups using a different method. In another recent preprint, T. Kawabe [K] has obtained a proof of Corollary 1.2 for countable discrete groups when the URS consists of amenable subgroups.

Acknowledgements. We are grateful to Adrien Le Boudec for useful discussions. We would also like to thank Uri Bader and Pierre-Emmanuel Caprace for indicating that [AG, Lemma I.1] allows to avoid the metrizability assumptions in [GW] in the definition of a stabilizer URS.

2. PROOF FOR DISCRETE GROUPS

If G is a discrete group, Theorem 1.1 is substantially simpler, therefore we prove this first.

Let Z be a discrete set. We denote by βZ its Stone–Čech compactification. Given a subset $W \subseteq Z$, the notation \overline{W} refers to the closure in βZ . Given a group action $G \curvearrowright Z$ and $g \in G$, we denote by $\text{Fix}_g(Z)$ the set of points fixed by g , and by $\text{Mov}_g(Z)$ its complement. The proof of the following lemma is inspired by Ellis’s theorem [E2] that the action $G \curvearrowright \beta G$ is free.

Lemma 2.1. *Let $G \curvearrowright Z$ be a group action on a discrete set Z . Then for every $g \in G$, we have $\text{Fix}_g(\beta Z) = \overline{\text{Fix}_g(Z)}$ and $\text{Mov}_g(\beta Z) = \overline{\text{Mov}_g(Z)}$. In particular, the stabilizer map $\text{Stab}: \beta Z \rightarrow \text{Sub}(G)$ is continuous.*

Proof. Clearly $\text{Fix}_g(\beta Z) \supseteq \overline{\text{Fix}_g(Z)}$. Moreover, $\beta Z = \text{Fix}_g(\beta Z) \sqcup \text{Mov}_g(\beta Z) = \overline{\text{Fix}_g(Z)} \cup \overline{\text{Mov}_g(Z)}$ (the second equality follows from the density of Z), and therefore $\text{Mov}_g(\beta Z) \subseteq \overline{\text{Mov}_g(Z)}$. Let us check the reverse inclusions. We can find a function $f: Z \rightarrow \{0, 1, 2\}$ with the property that for every $x \in \text{Mov}_g(Z)$, we have $|f(gx) - f(x)| \geq 1$ (such a function can be easily defined separately on every g -orbit). The function f extends to a function on βZ that we still denote by f . It follows that for every $\omega \in \overline{\text{Mov}_g(Z)}$, we have $|f(g\omega) - f(\omega)| \geq 1$, and therefore $\omega \in \text{Mov}_g(\beta Z)$, showing that $\overline{\text{Mov}_g(Z)} = \text{Mov}_g(\beta Z)$. This implies in particular that $\overline{\text{Mov}_g(Z)}$ and $\overline{\text{Fix}_g(Z)}$ are disjoint. The inclusion $\text{Fix}_g(\beta Z) \subseteq \overline{\text{Fix}_g(Z)}$ then also follows from the fact that $\beta Z = \overline{\text{Fix}_g(Z)} \cup \overline{\text{Mov}_g(Z)}$.

Finally, it is well-known that for discrete groups the continuity of the stabilizer map is equivalent to the fact that for every $g \in G$ the set $\text{Fix}_g(\beta Z)$ is clopen (see, e.g., [LBMB, Lemma 2.2]), which is a consequence of the first statement. \square

Given a collection of subgroups $A \subseteq \text{Sub}(G)$, we write $Z_A = \sqcup_{H \in A} G/H$ and endow it with the discrete topology. There is an obvious action $G \curvearrowright Z_A$, by letting G act separately on each coset space.

Proposition 2.2. *Let G be a discrete group and $\mathcal{H} \subseteq \text{Sub}(G)$ be a closed invariant subset. Let $A \subseteq \mathcal{H}$ be a subset such that the set of all conjugates of subgroups in A is dense in \mathcal{H} . Then the compact G -space $X = \beta Z_A$ verifies the conclusion of Theorem 1.1.*

Remark 2.3. Of course, one can choose $A = \mathcal{H}$. However, if \mathcal{H} is assumed to be a URS, then one can simply choose $A = \{H\}$ for any $H \in \mathcal{H}$, so that $X = \beta(G/H)$.

Proof. Continuity of the stabilizer map was already proved in Lemma 2.1. Moreover, the image of $Z_A \subseteq \beta Z_A$ is a dense subset of \mathcal{H} by the assumption on A . Since Z_A is dense in βZ_A , it follows that the image of βZ_A is precisely \mathcal{H} . \square

For the reduction to a metrizable space when G is countable discrete, we refer directly to the general case of a second countable locally compact group (cf. Proposition 3.6). However, we note that in this case, one can always choose a metrizable realization of the URS that is zero-dimensional.

3. PROOF FOR LOCALLY COMPACT GROUPS

Let G be a locally compact group. We will always see G as a uniform space endowed with the *right uniformity* whose entourages are

$$\mathcal{U}_V = \{(g_1, g_2) : \exists v \in V \ vg_1 = g_2\},$$

where V varies over symmetric neighborhoods of 1_G . (Note that some authors call this the *left uniformity* instead.)

A pseudometric d on G is called *right-invariant* if $d(g_1h, g_2h) = d(g_1, g_2)$ for all $g_1, g_2, h \in G$, and is said to be (right) *uniformly continuous* if it is uniformly continuous as a function $d: G \times G \rightarrow \mathbf{R}$. Note that every continuous, right-invariant pseudometric is uniformly continuous. Our first step is to extract from the proof of the Birkhoff–Kakutani metrization theorem what we need for the sequel.

Lemma 3.1. *Let $g \in G$, $g \neq 1_G$ and let U be a neighborhood of 1_G . Then there exists a right-invariant, continuous pseudometric d on G such that:*

- (i) $d \leq 8$;
- (ii) $1/2 \leq d(1_G, g) \leq 1$;
- (iii) *the d -ball of radius 4 around 1_G is relatively compact;*
- (iv) $\{x \in G : d(1_G, x) < 1/2\} \subseteq U$.

Proof. We follow the proof of the Birkhoff–Kakutani theorem from [B]. Without loss of generality, we may assume that U is *symmetric* ($U = U^{-1}$), relatively compact and that $g \notin U$. Let $V_0 = U \cup gU \cup (gU)^{-1}$, $V_{-1} = V_0^3$, $V_{-2} = V_{-1}^3$, $V_{-3} = G$; let V_1 be a symmetric neighborhood of 1_G such that $V_1^3 \subseteq U$, and for each $n \geq 1$, let V_{n+1} be a symmetric neighborhood of 1_G such that $V_{n+1}^3 \subseteq V_n$. Thus for all $n \geq -3$, V_n is symmetric and $V_{n+1}^3 \subseteq V_n$. Define $\rho: G^2 \rightarrow \mathbf{R}$ by

$$\rho(x, y) = \inf\{2^{-n} : xy^{-1} \in V_n\}$$

and $d: G^2 \rightarrow G$ by

$$d(x, y) = \inf\left\{\sum_{i=0}^{k-1} \rho(x_i, x_{i+1}) : x_0 = x, x_k = y, x_1, \dots, x_{k-1} \in G\right\}.$$

We have that ρ is symmetric, right-invariant and

$$\rho(x_0, x_1) \leq \epsilon \text{ and } \rho(x_1, x_2) \leq \epsilon \text{ and } \rho(x_2, x_3) \leq \epsilon \implies \rho(x_0, x_3) \leq 2\epsilon.$$

By [B, Lemma 6.2], d is a right-invariant pseudometric on G that satisfies

$$\frac{1}{2}\rho(x, y) \leq d(x, y) \leq \rho(x, y), \quad \text{for all } x, y \in G.$$

By the triangle inequality and right invariance, we have

$$|d(ux, vy) - d(x, y)| \leq d(ux, x) + d(vy, y) \leq \rho(u, 1_G) + \rho(v, 1_G),$$

showing that d is right uniformly continuous. Observe that ρ may not separate points and that is why we obtain a pseudometric rather than a metric.

We note that as $g \in V_0 \setminus V_1$, we have that $\rho(1_G, g) = 1$ and thus $1/2 \leq d(1_G, g) \leq 1$. Moreover, $\{x \in G : d(1_G, x) < 4\} \subseteq V_{-2}$ is relatively compact. Finally, if $x \notin V_1$, we have $\rho(1_G, x) \geq 1$ and thus $d(1_G, x) \geq 1/2$, proving that $\{x \in G : d(1_G, x) < 1/2\} \subseteq V_1 \subseteq U$. \square

Remark 3.2. If G is second countable, then by a result of Struble [S], there always exists a proper, right-invariant metric on G and in that case, one can use this metric instead of the pseudometric provided by Lemma 3.1 in what follows (with small modifications of the proof).

Given a closed subgroup $H \leq G$, we equip the homogeneous space G/H with the quotient of the right uniformity of G . Explicitly, its entourages are

$$\mathcal{U}_V = \{(g_1H, g_2H) : \exists v \in V \ vg_1H = g_2H\},$$

where V varies over symmetric neighborhoods of 1_G . If d is a right-invariant, continuous pseudometric on G , define d_H on G/H by

$$(3.1) \quad d_H(g_1H, g_2H) = \inf_{h \in H} d(g_1h, g_2).$$

Note that by right invariance, d_H is a pseudometric on G/H . Moreover, for every $g \in G$, we have $d_H(gxH, xH) \leq d(g, 1_G)$ which implies that d_H is uniformly continuous.

Given $g \in G$ and $V \ni 1_G$, we denote

$$\text{Mov}_g^V(G/H) = \{xH \in G/H : gxH \notin VxH\}.$$

The following lemma is adapted from the proof of Veech's theorem by Kechris, Pestov, and Todorćević [KPT, Appendix A].

Lemma 3.3. *Let $g \in G$ and $V \ni 1_G$ be open. Let $H \leq G$ be a closed subgroup. Then there exists $n \in \mathbf{N}$ and a uniformly continuous function $F: G/H \rightarrow \mathbf{R}^n$ with $\|F\|_\infty \leq 8$ such that*

$$\|F(gxH) - F(xH)\|_\infty \geq 1/4 \quad \text{for every } xH \in \text{Mov}_g^V(G/H).$$

Moreover, the dimension n of the target \mathbf{R}^n can be chosen to depend only on g and V but not on H .

Proof. Choose a metric d as in Lemma 3.1 (with $U = V$). Define d_H as in (3.1). Using Zorn's lemma, choose a subset $A \subseteq G/H$ which is maximal with the property

$$aH, bH \in A \text{ and } aH \neq bH \implies d_H(aH, bH) \geq 1/8.$$

Define a graph Γ with vertex set A where aH and bH are connected by an edge if and only if $d_H(aH, bH) < 3$.

We claim that Γ has bounded degree and that the bound on the degree does not depend on H . To see this, let b_1H, \dots, b_nH be distinct neighbors of aH . This means that there exist $h_1, \dots, h_n \in H$ such that $d(a, b_ih_i) < 3$ for every $i = 1, \dots, n$. Furthermore, by the definition of A , we have $d(b_ih_i, b_jh_j) \geq 1/8$ for every $i \neq j$. Since d is right-invariant, this implies that the elements $x_i = b_ih_ia^{-1}$ lie in the ball of radius 3 around 1_G and have distance at least $1/8$ between each other. It

follows that their cardinality does not exceed the size ℓ of a finite cover by balls of radius $1/16$ of the ball of radius 3 (which is relatively compact by Lemma 3.1). Therefore Γ has degree bounded by ℓ .

It follows that Γ can be colored with at most $n = \ell + 1$ colors in such a way that no two adjacent vertices have the same color. Let $A = A_1 \sqcup \dots \sqcup A_n$ be the resulting partition of the vertices. For every $i = 1, \dots, n$, let $f_i: G/H \rightarrow \mathbf{R}$ be given by $f_i(xH) = d_H(xH, A_i)$. Set $F = (f_1, \dots, f_n)$.

Consider $xH \in \text{Mov}_g^V(G/H)$ and note that condition (iv) in Lemma 3.1 implies that $d_H(xH, gxH) \geq 1/2$. By the definition of A , there exists a point $aH \in A$ such that $d_H(xH, aH) < 1/8$. Let i be such that $aH \in A_i$. Then $f_i(xH) < 1/8$.

Next we examine $f_i(gxH)$. Observe that

$$\begin{aligned} d_H(gxH, aH) &\leq d_H(gxH, xH) + d_H(xH, aH) \\ &\leq d(gx, x) + 1/8 \leq 9/8. \end{aligned}$$

We claim that aH is the closest point in A_i to gxH . Indeed, if another point in A_i were closer to xH , it would have to lie at a distance less than $18/8$ from aH , which is not possible because two points in A_i lie at distance at least 3 . Therefore

$$\begin{aligned} f_i(gxH) &= d_H(gxH, aH) \\ &\geq d_H(gxH, xH) - d_H(xH, aH) \\ &\geq 1/2 - 1/8 = 3/8. \end{aligned}$$

We deduce that

$$\|F(gxH) - F(xH)\|_\infty \geq |f_i(gxH) - f_i(xH)| \geq 3/8 - 1/8 = 1/4. \quad \square$$

We are now ready to prove Theorem 1.1. Let $\mathcal{H} \subseteq \text{Sub}(G)$ be a closed invariant subset. Let $A \subseteq \mathcal{H}$ be such that the union of the orbits of elements of A is dense in \mathcal{H} . Let $Z = \sqcup_{H \in A} G/H$, endowed with the disjoint union topology and uniform structure. For $g \in G$ and $V \ni 1_G$ open, we denote

$$\text{Mov}_g^V(Z) = \{z \in Z : gz \notin Vz\} = \sqcup_{H \in A} \text{Mov}_g^V(G/H).$$

As a consequence of the last sentence in Lemma 3.3 (stating that the dimension n of the codomain of F is uniform in H), if one is given g and V , the functions F obtained in Lemma 3.3 can be coalesced together to obtain a uniformly continuous function on Z , and therefore Lemma 3.3 remains valid for the uniform space Z . We record this in the next lemma.

Lemma 3.4. *Let $g \in G$ and $V \ni 1_G$ be open. Then there exists $n \in \mathbf{N}$ and a bounded, uniformly continuous function $F: Z \rightarrow \mathbf{R}^n$ such that*

$$\|F(gz) - F(z)\|_\infty \geq 1/4 \quad \text{for every } z \in \text{Mov}_g^V(Z).$$

Let $C_{\text{ub}}(Z)$ be the commutative C^* -algebra of bounded, uniformly continuous functions on Z and let $S(Z)$ be its Gelfand spectrum (this is often called the Samuel compactification of the uniform space Z).

Proposition 3.5. *The G -space $X = S(Z)$ verifies the conclusion of Theorem 1.1.*

Proof. Since Z is dense in X , it is enough to prove that for every $\omega \in X$ and every net $(z_i)_i \subseteq Z$ converging to ω , the stabilizers G_{z_i} converge to G_ω . Fix $\omega \in X$ and a net $(z_i) \subseteq Z$ with $z_i \rightarrow \omega$. By compactness, we may assume that G_{z_i} converges to some $K \in \mathcal{H}$. We have $K \leq G_\omega$.

Towards a contradiction, suppose that the inclusion is strict and let $g \in G_\omega \setminus K$. Let $V \ni 1_G$ be a compact, symmetric neighborhood of 1_G such that $Vg \cap K = \emptyset$. This defines an open condition for the Chabauty topology, so $g \notin VG_{z_i}$ for i large enough, i.e., $z_i \in \text{Mov}_g^V(Z)$. By Lemma 3.4, we can find a function $F: Z \rightarrow \mathbf{R}^n$

with the property that $\|F(gz_i) - F(z_i)\|_\infty \geq 1/4$ for all i large enough. Since F extends to X , passing to the limit, we get $\|F(g\omega) - F(\omega)\|_\infty \geq 1/4$, contradicting the fact that $g \in G_\omega$. Therefore $K = G_\omega$ and the stabilizer map is continuous as claimed.

That the image of Stab is equal to \mathcal{H} now follows from the fact that $\text{Stab}(Z)$ is a dense subset of \mathcal{H} . \square

It remains to prove the claim of the last sentence in the statement of Theorem 1.1.

Proposition 3.6. *Let X be the G -space constructed above. If G is second countable, then there exists a metrizable quotient Y of X such that Stab is continuous on Y and $\text{Stab}(Y) = \mathcal{H}$.*

Proof. Fix a countable basis \mathcal{B} at 1_G . Let $Z = \sqcup_{H \in \mathcal{A}} G/H$ as before. We will define the quotient Y as the Gelfand space of a separable G -invariant subalgebra \mathcal{A} of $C_{\text{ub}}(Z)$. Note that for the Stab map to be continuous on Y , we only need that Lemma 3.4 hold for \mathcal{A} , i.e.:

$$\forall V \in \mathcal{B} \forall g \in G \exists F \in \mathcal{A} \quad \|F(gz) - F(z)\|_\infty > 1/8 \quad \text{for all } z \in \text{Mov}_g^V(Z),$$

that is, the function $F = (f_1, \dots, f_n): Z \rightarrow \mathbf{R}^n$ can be chosen in such a way that $f_1, \dots, f_n \in \mathcal{A}$. Thus all we need to show is that for a fixed $V \in \mathcal{B}$, there is a countable collection \mathcal{A}_V of functions $F: Z \rightarrow \mathbf{R}^n$ such that

$$\forall g \in G \exists F \in \mathcal{A}_V \forall z \in Z \quad gz \in Vz \quad \text{or} \quad \|F(gz) - F(z)\|_\infty > 1/8.$$

Provided that this is done, we can take \mathcal{A} to be the smallest G -invariant, closed subalgebra that contains $\bigcup_{V \in \mathcal{B}} \mathcal{A}_V$, which is still separable.

Lemma 3.4 and uniform continuity imply that for every $g \in G$, there exist $F: Z \rightarrow \mathbf{R}^n$ and an open $U \ni g$ such that

$$\forall g' \in U \forall z \in Z \quad g'z \in Vz \quad \text{or} \quad \|F(g'z) - F(z)\|_\infty > 1/8.$$

Now the fact that G is Lindelöf implies that we can find a countable collection of functions F that works for all g . \square

4. UNIVERSAL MINIMAL FLOW RELATIVE TO A URS

4.1. Existence and uniqueness. If \mathcal{H} and \mathcal{K} are URSs of G , we write $\mathcal{H} \preceq \mathcal{K}$ if for all $H \in \mathcal{H}$, there exists $K \in \mathcal{K}$ such that $H \leq K$. This relation is a partial order on the set of URSs of G (see [LBMB, Corollary 2.15]; the proof given there for countable groups extends easily to locally compact groups), however we shall not use this fact.

Definition 4.1. Let G be a group, $G \curvearrowright X$ be a minimal action on a compact space X , and let \mathcal{H} be a URS of G . We will say that X is *subordinate* to \mathcal{H} if $\mathcal{H} \preceq \mathcal{S}_G(X)$.

Recall that given two compact G -spaces X and Y , we say that X *factors onto* Y if there exists a continuous, surjective, G -equivariant map $X \rightarrow Y$. Given a collection \mathcal{E} of compact G -spaces, we say that $X \in \mathcal{E}$ is *universal* for \mathcal{E} if it factors onto all elements of \mathcal{E} .

The goal of this section is to establish the following theorem.

Theorem 4.2. *For every URS \mathcal{H} of G , there exists a minimal G -space $M(G, \mathcal{H})$, unique up to isomorphism, which is subordinate to \mathcal{H} and is universal for minimal G -spaces subordinate to \mathcal{H} .*

Definition 4.3. The space $M(G, \mathcal{H})$ will be called the *universal minimal flow* of G relative to \mathcal{H} .

If $\mathcal{H} = \{\{1_G\}\}$, then $M(G, \mathcal{H})$ is just the usual universal minimal flow of G .

The existence is easy. Let $H \in \mathcal{H}$ be arbitrary and recall that $S(G/H)$ denotes the Samuel compactification of G/H . Let $M \subseteq S(G/H)$ be a minimal subset. Then M verifies the universal property. Indeed, let $G \curvearrowright X$ be a minimal G space subordinate to \mathcal{H} . Then there exists a point $x \in X$ such that H stabilizes x . The orbital map $G \rightarrow X, g \mapsto g \cdot x$ descends to a uniformly continuous map $G/H \rightarrow X$, which extends to a G -map $S(G/H) \rightarrow X$, and taking the restriction to M shows that M factors onto X . We have already proven that the collection of stabilizers of $G \curvearrowright M$ is equal to \mathcal{H} ; in particular, M is subordinate to \mathcal{H} .

Our next goal is to check uniqueness. For this, it is enough to prove that M is *coalescent*, i.e., that every continuous G -equivariant map $M \rightarrow M$ is a homeomorphism. For the usual (non-relative) universal minimal flow of G , this is a result of Ellis [E2]. Our proof is close to the exposition by Uspenskij [U] of Ellis's theorem. The proof of this result is based on the fact that $S(G)$ carries a natural semigroup structure. The main difference is that in our case, $S(G/H)$ does not carry such a structure; however, we can find a semigroup inside $S(G/H)$ that is sufficient for our purposes.

Let $\text{Fix}_H(M)$ be the set of points in M fixed by H . Observe that for every $\omega \in \text{Fix}_H(M)$, the orbital map $G/H \rightarrow G \cdot \omega$ extends to a continuous equivariant map $r_\omega: S(G/H) \rightarrow M$, which is moreover the unique G -map $S(G/H) \rightarrow S(G/H)$ sending H to ω . Hence, we get a map $S(G/H) \times \text{Fix}_H(M) \rightarrow M$ continuous in the first variable.

Lemma 4.4. *For every $\omega \in \text{Fix}_H(M)$, we have $r_\omega(\text{Fix}_H(M)) \subseteq \text{Fix}_H(M)$. In particular, $\text{Fix}_H(M)$ is a right-topological semigroup under the operation $\text{Fix}_H(M) \times \text{Fix}_H(M) \rightarrow \text{Fix}_H(M), (\eta, \omega) \mapsto \eta\omega := r_\omega(\eta)$.*

Proof. This is obvious because the map r_ω is G -equivariant. \square

Since $\text{Fix}_H(M)$ is a compact, right-topological semigroup, by a well-known theorem of Ellis, $\text{Fix}_H(M)$ contains idempotent elements.

Lemma 4.5. *Let $\iota \in \text{Fix}_H(M)$ be an idempotent. Then the map $r_\iota: S(G/H) \rightarrow M$ is a retraction of $S(G/H)$ onto M .*

Proof. We need to prove that $r_\iota|_M = \text{id}$. Since $r_\iota(\iota) = \iota^2 = \iota$, by G -equivariance, r_ι is the identity on the orbit of ι , which is dense in M by minimality, whence the conclusion. \square

Lemma 4.6. *Every continuous G -map $M \rightarrow M$ is of the form r_ω for some $\omega \in \text{Fix}_H(M)$.*

Proof. Let $f: M \rightarrow M$ be a continuous G -map. Let $\iota \in \text{Fix}_H(M)$ be an idempotent. Consider $f \circ r_\iota: S(G/H) \rightarrow M$. As this map is continuous and equivariant, we have $f \circ r_\iota = r_\omega$ for $\omega = f(\iota)$. Since $r_\iota|_M = \text{id}$, this implies that $f = r_\omega$. \square

Proposition 4.7. *M is coalescent.*

Proof. Let $f: M \rightarrow M$ be a continuous G -map. We need to show that f is injective. By equivariance, we have $f(\text{Fix}_H(M)) \subseteq \text{Fix}_H(M)$. By Lemma 4.6, there exists $\omega \in \text{Fix}_H(M)$ such that $f = r_\omega$. Therefore $f(\text{Fix}_H(M)) = \text{Fix}_H(M)\omega$ is a compact left ideal of $\text{Fix}_H(M)$ and thus a compact subsemigroup of $\text{Fix}_H(M)$. By Ellis's theorem, there exists an idempotent $\iota \in \text{Fix}_H(M)\omega$. Let $\eta \in \text{Fix}_H(M)$ be such that $f(\eta) = \eta\omega = \iota$. Let $g = r_\eta$. Now by Lemma 4.5, for all $x \in M$,

$$(f \circ g)(x) = r_\omega(r_\eta(x)) = x\eta\omega = x\iota = r_\iota(x) = x.$$

The map g , being continuous and equivariant, is surjective by minimality. Since $f \circ g = \text{id}$, it follows that f is injective. \square

4.2. Metrizability of $M(G, \mathcal{H})$. It is a natural question for which pairs (G, \mathcal{H}) the relative universal minimal flow $M(G, \mathcal{H})$ can be identified with a more familiar, concrete G -space. A case in which this can be done is when the URS \mathcal{H} contains a cocompact subgroup H (and thus is necessarily a single compact conjugacy class). In this case, $M(G, \mathcal{H})$ can be identified with the homogeneous space G/H . The following proposition says that there is little hope beyond this case.

Proposition 4.8. *Let G be a locally compact second countable group and let \mathcal{H} be a URS of G . Then $M(G, \mathcal{H})$ is metrizable iff \mathcal{H} contains a cocompact subgroup.*

Proof. The “if” direction is clear. For the other, suppose that $M(G, \mathcal{H})$ is metrizable. Following the argument for the proof of Theorem 1.2 in [BYMT], we conclude that $M(G, \mathcal{H})$ contains a G_δ orbit $G \cdot x_0$. As G is σ -compact, the orbit $G \cdot x_0$ is also F_σ , implying that its complement is G_δ . If the complement is non-empty, it must be dense by minimality, contradicting the Baire category theorem. Thus the action $G \curvearrowright M(G, \mathcal{H})$ is transitive and if we put $H = G_{x_0}$, Effros’s theorem (see, e.g., [H, Theorem 7.12]) implies that H is cocompact. As a consequence of Theorem 1.1, the point stabilizers of $G \curvearrowright M(G, \mathcal{H})$ are precisely the elements of \mathcal{H} . Therefore \mathcal{H} contains a cocompact subgroup as claimed. \square

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